CyberChallenge.IT 2023 - Test
Commented solutions

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1 Question 1

1.1 Question

Abibbo, Babibbo and Cabibbo compete on the challenges on the training platform. Given a challenge, each of them has a probability $p = \frac{1}{2}$ of solving it. They take turns in opening the challenges: if Abibbo solves its challenge, he wins, otherwise it's Babibbo's turn; if Babibbo solves its challenge, he wins, otherwise it's Cabibbo's turn; if Cabibbo solves its challenge, he wins, otherwise they start again from Abibbo. Find the probability that Babibbo wins.

1.2 Answers

(A) 2/7

(B) 1/8

(C) 4/7

(D) 1/2

1.3 Proposed solution

The correct answer is `(A) 2/7`.

Babibbo wins if:

- Abibbo loses in the first round and Babibbo wins in the second one
- Abibbo, Babibbo and Cabibbo lose in the first 4 rounds and Babibbo wins in the fifth one
- Abibbo, Babibbo and Cabibbo lose in the first 7 rounds and Babibbo wins in the eighth one
- and so on

Let us notice that the same situation repeat itself every 3 rounds, so that we can write

$$p_B = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot p_B$$

leading to

$$p_B = \frac{2}{7}.$$

2 Question 2

2.1 Question

Gaspare is writing personalized emails to the ten Team Italy finalists, but he did a mess with the addresses and he doesn't know to which person every address corresponds. In how many ways can he send the emails such that at least two finalists will receive the wrong email?

2.2 Answers

(A) 3628799

(B) 1814400

(C) 3628800

(D) 1814399

2.3 Proposed solution

The correct answer is `(A) 3628799`.

Notice that there are $10! = 3628800$ ways to send the emails. Furthermore, it is impossible that exactly one email is sent to the wrong person, hence we have to subtract just the number of ways in which all the emails are correctly sent, which is 1.
The requested quantity is $10! - 1 = 3628799$.

3 Question 3

3.1 Question

An array contains the numbers $[1, 3, 5, ..., 2047]$ (the odd integers from 1 to 2047 included). Fixed a positive integer k, a move consists in choosing k elements from the array, applying the bitwise AND between the chosen elements and subtracting this last value from those k integers. For how many values of k it is possible to reach in a finite number of moves an array with all elements equal to 0?

Note: the bitwise AND, also denoted by &, is the bit-by-bit operation defined by the following truth table, $0\&0 = 0, 1\&0 = 0, 0\&1 = 0, 1\&1 = 1$.

3.2 Answers

(A) 10

(B) 1

(C) 7

(D) 12

3.3 Proposed solution

The correct answer is `(A) 10`.

Notice that a move is "cancelling" out some 1s from the binary representation of the elements. Thus, we want k to divide the total number of 1s, for each position, over all the elements of the array. The least significant bit is always 1 (1024 occurrences), while for all the other positions there are half of the elements with the corresponding bit set to 0 and the other half with the corresponding bit set to 1 (512 occurrences).
Therefore, k must divide $\gcd(1024, 512) = 512$, and 512 has 10 divisors $(1, 2, 4, \ldots, 256, 512)$.

4 Question 4

4.1 Question

A solid cube of side length 1 is removed from each corner of a solid cube of side length 3. How many edges does the remaining solid have?

4.2 Answers

(A) 84

(B) 36

(C) 60

(D) 108

4.3 Proposed solution

The correct answer is `(A) 84`.

A simple cube has 12 edges. The removal of each little cube introduces 9 more edges, and the number of removed cubes is 9. Thus, the number of edges in the final solid is $12 + 8 \cdot 9 = 84$.

5 Question 5

5.1 Question

Alice has two discs with a number written on one of their two sides: disc A has the number 11, while disc B has 9. What numbers have to be printed on the other side of each disc so that their faces, one of A and one of B, add to 19, 20, 21, 22 (the first number is printed on disc A, the second on disc B) and that no number on the discs is duplicated?

5.2 Answers

(A) 13, 8

(B) 10, 11

(C) 10, 8

(D) 8, 13

5.3 Proposed solution

The correct answer is `(A) 13, 8`.

If disc A has the numbers 11 and 13, and disc B has the numbers 9 and 8, we can write (the first number is from disc A, while the second from disc B):

- $19 = 11 + 8$
- $20 = 11 + 9$
- $21 = 13 + 8$
- $22 = 13 + 9$

The answer `10, 11`, although makes all the required sums possible, is invalid because the number 11 is duplicated.

6 Question 6

6.1 Question

300 people will participate in OliCyber.IT, CyberChallenge.IT or both. $\frac{1}{6}$ of those who will participate in Olicyber.IT will participate also in CyberChallenge.IT and $\frac{1}{10}$ of those who will participate in CyberChallenge.IT will participate also in OliCyber.IT. How many people will participate only in CyberChallenge.IT?

6.2 Answers

(A) 180

(B) 200

(C) 150

(D) 220

6.3 Proposed solution

The correct answer is `(A) 180`.

In order to keep the given proportions, for each person that will participate in both projects there are 5 people that will participate only in OliCyber.IT and 9 only in CyberChallenge.IT, for a total of 15 people.
There will be 300 participants, so we have to multiply those numbers by $\frac{300}{15} = 20$, resulting in 20 people doing both projects, 100 doing only OliCyber.IT and 180 doing only CyberChallenge.IT.

7 Question 7

7.1 Question

On an island, there are knights, who always tell the truth, and impostors, who always lie. A, B, C and D are the inhabitants of the island and are talking about the number of knights. They say:

A: There are 0, 1 or 3 knights.

B: There are 1, 2 or 3 knights.

C: Excluding me, there are 0, 1 or 3 knights.

D: Excluding me, there are 1, 2 or 3 knights.

How many knights live on the island?

7.2 Answers

(A) 2

(B) 0

(C) 1

(D) 3

7.3 Proposed solution

The correct answer is `(A) 2`.

We can test each number.
If none of them is a knight, then A tells the truth, which is a contradiction.
If one of them is a knight, then both A and B are telling the truth, which is a contradiction.
If two of them are knights, then A lies and B tells the truth; if C lies and D tells the truth, there is no contradiction.
If three of them are knights, then A and B tell the truth, so D tells the truth, but at this point also C tells the truth, contradiction.
The answer is therefore 2.

8 Question 8

8.1 Question

Bruno wants to train himself for the finals of CyberChallenge.IT. In order to do so, he solves either two or three challenges each day, tossing a coin. But since solving challenges is difficult, if he solves three challenges in three consecutive days, the next day he does not solve any challenge.
What is the probability that the number of challenges solved by Bruno in the first 5 days is exactly 11?

8.2 Answers

(A) 9/32

(B) 7/32

(C) 1/4

(D) 7/29

8.3 Proposed solution

The correct answer is `(A) 9/32`.

There are 7 possible configurations that work:

- `23330`, that happens with probability $\frac{1}{16}$
- `33302`, that happens with probability $\frac{1}{16}$
- `22223`, `22232`, `22322`, `23222`, `32222`, each happening with probability $\frac{1}{32}$

The total probability is therefore $\frac{1}{16} + \frac{1}{16} + 5 \cdot \frac{1}{32} = \frac{9}{32}$.

9 Question 9

9.1 Question

Lorenzo and Mattia are playing a game at the Team Italy camp. They have a bag with n ping pong balls and, one at a time and starting from Lorenzo, remove some balls from the bag. More specifically, if there are x balls left, they can remove either $s(x)$ or $x - s(x)$ balls, being $s(x)$ the sum of the digits of x. The player who removes the last ball wins.
For how many values of n, between 1998 and 2023 included, Lorenzo has a winning strategy?

9.2 Answers

(A) 17

(B) 0

(C) 19

(D) 26

9.3 Proposed solution

The correct answer is `(A) 17`.

Let us define an integer n *good* if the player receiving n balls in the bag has a winning strategy.
The integers between 1 and 9 included are good. For $10 \leq n \leq 18$, we are forced to leave the bag with a good number of balls, thus these values of n are not good.
Notice that 1998 is not good: we can leave either 27 or 1971, from which the adversary can leave us with 18, which is not good. 1999 is not good as well. Instead, 2000 is good, since we can leave 1998 balls. Following the same reasoning, all the values up to 2009 are good.
The values $2010 \leq n \leq 2016$ are not good since the can be replaced with a single digit number or 2007, and they are both good.
With the same reasoning, we find that the values $2017 \leq n \leq 2023$ are good, since we can leave the adversary with a value that is not good.
Therefore, the number of good values is the total number of values, 26, minus the number of non good values, which is $2 + 7 = 9$, for a total of $26 - 9 = 17$.

10 Question 10

10.1 Question

Carlo leaked the formula used by the organizers to grade the programming test. It is a function $f(x)$ such that if you solve x problems, you score $f(f(f(f(x))))^2$ points. Moreover, it is known that $f(f(x)) = \sqrt{x^2 + 15}$.
If Carlo solved 7 problems, how many points did he get?

10.2 Answers

(A) 79

(B) 64

(C) 51

(D) 40

10.3 Proposed solution

The correct answer is `(A) 79`.

If $f(f(x)) = \sqrt{x^2 + 15}$, then $f(f(7)) = \sqrt{7^2 + 15} = \sqrt{64} = 8$. The final score is

$$f(f(f(f(7))))^2 = f(f(8))^2 = \sqrt{8^2 + 15}^2 = \sqrt{79}^2 = 79.$$

11 Question 11

11.1 Question

Consider the following functions:

```
function f(x){
    if(x <= 0){
        return 1
    }
    return (25+g(x-2))*2
}

function g(x){
    if(x <= 0){
        return 1
    }
    return (4+f(x-5))*5
}
```

How many digits does `f(`10^{13}`)` have?

11.2 Answers

(A) 1428571428574

(B) 1773001770432

(C) 1505234295327

(D) 1686761608829

11.3 Proposed solution

The correct answer is `(A) 1428571428574`.

Let us rewrite

$$f(x) = 50 + 2g(x-2) = 50 + 2(20 + 5f(x-7)) = 90 + 10f(x-7).$$

Given that $10^{13} = 1428571428571 \cdot 7 + 3$, we are interested in $f(3) = 100$. Then, we can find that:

- $f(10) = 90 + 10f(3) \ = 1090$
- $f(17) = 90 + 10f(10) = 10990$
- $f(24) = 90 + 10f(17) = 109990$
- $f(31) = 90 + 10f(24) = 1099990$

Intuitively, we can say that the number of digits of $f(3+7k)$ is $k+3$, from which $f(10^{13}) = f(1428571428571 \cdot 7+3)$ have $1428571428571 + 3 = 1428571428574$ digits.
Otherwise, we can prove the same using the closed formula. In fact, it holds the following:

$$f(x + 7k) = (10 + f(x)) \cdot 10^k - 10.$$

12 Question 12

12.1 Question

Consider the following functions:

```
function p(v, a, n){
    x = a
    int flag[n]
    for(i=0; i<n; i++){
        flag[x-1] = v[i]
        x = (a*x) % (n+1)
    }
    return flag
}

function f(x){
    v = [0,1,2,3,4,5,6,7,8,9]
    for(i=0; i<x; i++){
        v = p(v, 7, 10)
    }
    return v
}
```

What is the result of `f(3133333333337)`?

12.2 Answers

(A) `[9, 2, 3, 5, 1, 6, 0, 8, 7, 4]`

(B) `[2, 6, 0, 9, 5, 4, 1, 7, 8, 3]`

(C) `[6, 4, 1, 2, 9, 3, 5, 8, 7, 0]`

(D) `[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]`

12.3 Proposed solution

The correct answer is `(A) [9, 2, 3, 5, 1, 6, 0, 8, 7, 4]`.

The code of function `p` is returning a permutation of vector `v` described by the variable `x`, while the function `f` is iterating that permutation.

We can compute the first iterations looking for the initial vector: that would mean that we found the maximum length of the permutation cycle[1].

```
f(1) = [9, 2, 3, 5, 1, 6, 0, 8, 7, 4]
f(2) = [4, 3, 5, 6, 2, 0, 9, 7, 8, 1]
f(3) = [1, 5, 6, 0, 3, 9, 4, 8, 7, 2]
f(4) = [2, 6, 0, 9, 5, 4, 1, 7, 8, 3]
f(5) = [3, 0, 9, 4, 6, 1, 2, 8, 7, 5]
f(6) = [5, 9, 4, 1, 0, 2, 3, 7, 8, 6]
f(7) = [6, 4, 1, 2, 9, 3, 5, 8, 7, 0]
f(8) = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
```

So the permutation is repeating itself every 8 iterations. At this point, we can write
`f(3133333333337) = f(391666666667*8+1) = f(1) = [9, 2, 3, 5, 1, 6, 0, 8, 7, 4]`.

[1]See `https://en.wikipedia.org/wiki/Cyclic_permutation`, that also suggests a more efficient method to compute the cycle length

13 Question 13

13.1 Question

Consider the following functions:

```
function f(n){
    if(n == 0){
        return 1
    }
    return n*f(n-1)
}

function v(n){
    r = 0
    while((n > 0) && !(n % 10)){
        r = r+1
        n = n/10
    }
    return r
}

function m(x){
    for(i=0; i<x; i++){
        if(v(f(i)) == v(f(i+1))){
            print("A")
        }
    }
}
```

How long is the output of `m(314159265)`?

13.2 Answers

(A) 251327412

(B) 62831853

(C) 141405835

(D) 70239259

13.3 Proposed solution

The correct answer is `(A) 251327412`.

Analyzing the code, we may find that the function `f(n)` is computing the factorial of n, while the function `v(n)` is computing the largest power of ten that divides n.
Given that in the factorization of $n!$ the number of factors 2 is always greater than or equal than the number of factors 5, we can consider this last quantity, in order to evaluate `v(f(n))`. More specifically, we may notice that `v(f(i)) != v(f(i+1))` when `i+1` is a multiple of 5.
Therefore, an `"A"` is printed $n - \left\lfloor \frac{n}{5} \right\rfloor$ times; substituting the value $n = 314159265$ we obtain 251327412.

14 Question 14

14.1 Question

Consider the following function:

```
function f(x){
    if(x == 0){
        return 1
    }
    if(x == 1){
        return 1
    }
    return 4*f(x-1)-8*f(x-2)
}
```

What is the value of `f(98765432)`?

14.2 Answers

(A) $2^{148148148}$

(B) $2^{121297759}$

(C) $3 \cdot 2^{167194010}$

(D) $2^{118020608}$

14.3 Proposed solution

The correct answer is `(A)` $2^{148148148}$.

Notice that, for $x \geq 0$ we have:

$$\begin{aligned} f(x+4) &= 4f(x+3) - 8f(x+2) \\ &= 16f(x+2) - 32f(x+1) - 8f(x+2) = 8f(x+2) - 32f(x+1) \\ &= 32f(x+1) - 64f(x) - 32f(x+1) = -64f(x) \end{aligned}$$

This means that every 4 steps, the value returned by the function is multiplied by -64, and every 8 the value is multiplied by $4096 = 2^{12}$. Let us notice that $98765432 = 8 \cdot 12345679$, this means that

$$f(98765432) = (2^{12})^{12345679} \cdot f(0) = 2^{12 \cdot 12345679} = 2^{148148148}.$$

15 Question 15

15.1 Question

Consider a sorted array of 100 integers. How many elements need to be checked at most in order to determine if a specific value is contained in the array?

15.2 Answers

(A) 7

(B) 99

(C) 6

(D) 100

15.3 Proposed solution

The correct answer is `(A) 7`.

Since the array is sorted, in order to efficiently search in it we can use a binary search[2] In the worst case scenario, for an array long n this algorithm needs $\lfloor \log_2(n) + 1 \rfloor$ iterations, with one comparison per iteration. For $n = 100$, as in our case, the algorithm needs at most $\lfloor \log_2(100) + 1 \rfloor = 7$.

[2] See `https://en.wikipedia.org/wiki/Binary_search_algorithm`.